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ON THE BASICITY FROM EXPONENTS IN LEBESGUE SPACES WITH VARIABLE EXPONENT

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Abstract. In the paper we consider the systems of exponents $\{\exp i\,(n-\alpha sign n)\,t\}_{n\in Z}$, $1\cup\{\exp i\,(n-\alpha sign n)\,t\}_{n\neq 0}$, *cosines* $\{\cos(n-\alpha)\,t\}_{n\geq 0}$ $(1\cup\{\cos(n-\alpha)\,t\}_{n\geq 1})$ and *sines* $\{\sin(n-\alpha)\,t\}_{n\geq 1}$. The basis properties of these systems are completely studied in the spaces L_{p_t} with variable exponent $p\,(t)$.

Keywords: a system of exponents, basicity, completeness, minimality, Lebesgue space with variable exponent.

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1. Introduction

The paper studies the basicity of the system of exponents

$$\left\{e^{i(n-\alpha\cdot sign n)t}\right\}_{n\in Z}, \tag{1}$$

$$1\cup\left\{e^{i(n-\alpha\cdot sign n)t}\right\}_{n\neq 0} \tag{2}$$

in Lebesgue spaces of functions with variable exponent $p\,(t)$, denoted as L_{p_t}, where $\alpha\in C$ is a complex parameter, Z is a set of integers. Systems (1),(2) are model systems for studying spectral properties of some differential operators. They are obtained from ordinary system of exponents by linear perturbation. The well-known mathematicians as Paley-Wiener [15], N. Levinson [14] and others were the first who appealed to study basis properties of these systems. Basis properties of systems (1),(2) were completely studied in Lebesgue ordinary spaces $L_p\,(p\,(t)\equiv const)$. Relatively these problems one can consider the papers [1,2,7,13] . Recently, in connection with consideration of some concrete problems of mechanics and mathematical physics,(see [12,16]), interest to studying these or other problems in the spaces L_{p_t} or $W^k_{p_t}$ increases.

In the present paper we study basicity of systems (1),(2) in $L_{p_t}\equiv L_{p_t}(-\pi,\pi)$ under definite conditions on the function $p:[-\pi,\pi]\to[1,+\infty)$.

2. Necessary notation and facts

Let $p:[-\pi,\pi]\to[1,+\infty)$ be some function measurable by Lebesgue. By $\mathcal{L}_0$ we denote a class of all measurable on $[-\pi,\pi]$ (with respect to Lebesgue measure) functions. Accept the denotation

$$I_p\,(f)\stackrel{def}{\equiv}\int_{-\pi}^{\pi}|f\,(t)|^{p(t)}\,dt.$$

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Let $\mathcal{L} \equiv \{f \in \mathcal{L}_0 : I_p(f) < +\infty\}$. With respect to ordinary linear operations of addition of functions and multiplication by a number, for $p^+ = \sup\limits_{[-\pi,\pi]} vrai\, p(t) < +\infty$, $\mathcal{L}$ turns into a linear space. By the norm

$$\|f\|_{p_t} \overset{def}{\equiv} \inf\left\{\lambda > 0 : I_p\left(\frac{f}{\lambda}\right) \leq 1\right\}$$

$\mathcal{L}$ is a Banach space and we denote it by L_{p_t}.

Denote

$$H^{\ln} \overset{def}{\equiv} \left\{p : \exists C > 0;\ \forall t_1, t_2 \in [-\pi,\pi], |t_1 - t_2| \leq \frac{1}{2} \Longrightarrow \right.$$

$$\left. \Longrightarrow |p(t_1) - p(t_2)| \leq \frac{C}{-ln|t_1 - t_2|}\right\}.$$

Everywhere $q(t)$ denotes a conjugated to $p(t)$ function: $\dfrac{1}{p(t)} + \dfrac{1}{q(t)} \equiv 1$. Accept $p^- = \inf\limits_{[-\pi,\pi]} vrai\, p(t)$, $p^\pi = \max\{p(\pi); p(-\pi)\}$, $p_\pi = \min\{p(\pi); p(-\pi)\}$. It holds Holder's generalized inequality

$$\int_{-\pi}^{\pi} |f(t)\, g(t)|\, dt \leq C\left(p^-; p^+\right) \|f\|_{p_t} \cdot \|g\|_{q_t},$$

where $C(p^-; p^+) = 1 + \dfrac{1}{p^-} - \dfrac{1}{p^+}$.

The following property follows directly from definition.

Property *A. If $|f(t)| \leq |g(t)|$ a.e. on $(-\pi,\pi)$, then $\|f\|_{p_t} \leq \|g\|_{p_t}$.*

We easily prove the following

Statement 1. *Let $p \in H^{\ln}, p(t) > 0$, $\forall t \in [-\pi,\pi]$ and $\{\alpha_i\}_1^m \subset R$ (R is a real axis). The function $\omega(t) = \prod\limits_{i=1}^{m} |t - t_i|^{\alpha_i}$ belongs to the space L_{p_t}, if $\alpha_i > -\dfrac{1}{p(t_i)}$, $\forall i = \overline{1,m}$; where $\{t_i\}_1^m \subset [-\pi,\pi]$, $t_i \neq t_j$ for $i \neq j$.*

In sequel, we'll need the following facts.

Property *B* **[16].** *If $p(t) : 1 < p^- \leq p^+ < +\infty$, the class $C_0^\infty(-\pi,\pi)$ (finite and infinitely differentiable) is everywhere dense in L_{p_t}.*

By S we denote a singular integral:

$$S(f) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\tau)}{\tau - t} d\tau,\ t \in \Gamma,$$

where $\Gamma \subset C$ is some piecewise-Holder curve on C (C is a complex plane).

Let $\rho : [-\pi,\pi] \to [1,+\infty)$ be some weight function. Determine a weight class $L_{p_t,\rho_t} : L_{p_t,\rho_t} \overset{def}{\equiv} \{f : \rho \cdot f \in L_{p_t}\}$ with the norm: $\|f\|_{p_t,\rho_t} \overset{def}{\equiv} \|\rho f\|_{p_t}$.

The following statement was proved in the paper [11].

Statement [11]. *Let $p \in H^{\ln}, 1 < p^-$ and $p(t) = \prod\limits_{k=1}^{m} |t - \tau_k|^{\alpha_k}$, where $\{\tau_k\}_1^m \subset [-\pi,\pi]$, $\tau_i \neq \tau_j$ for $i \neq j$. Then a singular operator S boundedly acts from L_{p_t,ρ_t} to L_{p_t,ρ_t}* if

$$-\frac{1}{p(\tau_k)} < \alpha_k < \frac{1}{q(\tau_k)},\ k = \overline{1,m}$$

are fulfilled.

The following classes of analytic functions play an important part while establishing basicity.

3. Hardy classes with variable exponent

These classes were considered in the papers [10,3]. Let $U \equiv \{z : |z| < 1\}$ be a unique ball on a complex plane and $\Gamma = \partial U$ be a unit circle. For a function $u(z)$ harmonic in U we accept

$$\|u\|_{p_t} \equiv \sup_{0<r<1} \left\| u\left(re^{it}\right) \right\|_{p_t},$$

where $p : [-\pi, \pi] \to [1, +\infty)$ is some measurable function. Denote

$$h_{p_t} \equiv \left\{ u : \Delta u = 0 \text{ in } U \text{ and } \|u\|_{p_t} < +\infty \right\}.$$

The continuous imbeddings $h_+ \hookrightarrow h_{p_t} \hookrightarrow h_{p^-}$ are true. The following theorem is valid.

Theorem [3]. *Let $1 < p^- \leq p^+ < +\infty$. If*

$$u \in h_{p_t}, \text{ then } \exists f \in L_{p_t} : \ u\left(re^{i\theta}\right) = \frac{1}{2\pi} \int_{-\pi}^{\pi} p_r(\theta - t) f(t)\, dt, \tag{3}$$

where $p_r(\alpha) = \dfrac{1 - r^2}{1 + r^2 - 2r\cos\alpha}$ is a Poisson kernel. Vice-versa, if $f \in L_{p_t}$, $p \in H^{\ln}$, then (3) belongs to h_{p_t}.

The Hardy class $H^+_{p_t}$ is introduced in the similar way

$$H^+_{p_t} :\equiv \left\{ f : f \text{ is analytic in } U \text{ and } \|f\|_{H^+_{p_t}} < +\infty \right\},$$

where $\|f\|_{H^+_{p_t}} = \sup\limits_{0<r<1} \left\| f\left(re^{it}\right) \right\|_{p_t}$.

It is easy to see that $f \in H^+_{p_t} \Longleftrightarrow Ref;\ Imf \in h_{p_t}$, where $Re\, z; Imz$ are real and imaginary parts of z, respectively.

Using the previous theorem we can easily prove the following refined variant of theorem 5 of the paper [10].

Theorem [10]. *Let $p \in H^{\ln}$ and $p^- > 1$. Then*

$$F \in H^+_{p_t} \Longleftrightarrow \exists f \in L_{p_t} : \ F(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} f(t)\, dt}{e^{it} - z}.$$

Analogy of Smirnov's known theorem is also valid.

Theorem [9]. *Let $p_i(t) : 0 < p_i^- \leq p_i^+ < +\infty,\ i = 1,2;\ p_1(t) \leq p_2(t)$, a.e. on $[-\pi, \pi]$ be measurable function, $F \in H^+_{p_t}$; $p_2 \in H^{\ln}$ and $p_2^- > 1$. Then, if $F^+ \in L_{p_{2t}} \Longrightarrow F \in H^+_{p_{2t}}$.*

Let's define the class ${}_m H^-_{p_t}$ of analytic outside of a unit circle functions of order $\leq m$ at infinity. Let $f(z)$ be a function analytic in $C \backslash \overline{U}$ $\left(\overline{U} = U \cup \Gamma\right)$, having a finite order $\leq m$ at infinity, i.e. $f(z) = f_1(z) + f_2(z)$, where $f_1(z)$ is a polynomial of power $\leq m$, $f_2(z)$ is a tame part of expansion of $f(z)$ in Lorentz series in the vicinity of a point at infinity. If the function $\varphi(z) \equiv \overline{f_2\left(\dfrac{1}{\overline{z}}\right)}$ ($\overline{(\cdot)}$ is a complex conjugation) belongs to the class $H^+_{p_t}$, we'll say that the function $f(z)$ belongs to the class ${}_m H^-_{p_t}$.

4. Riemann's problem in the classes $H^{\pm}_{p_t}$

Let a complex valued function $G(t)$ on $[-\pi, \pi]$ satisfy the conditions:

1) $|G|^{\pm 1} \in L_\infty$;

2) the argument $\theta(t) \equiv \arg G(t)$ has an expansion of the form $\theta(t) = \theta_0(t) + \theta_1(t)$, where $\theta_0(t) \in C[-\pi, \pi]$; $\theta_1(t)$ is a bounded variation function on $[-\pi, \pi]$;

3) $\theta_1(t)$ has a finite number of discontinuity points $\{s_k\}_1^r : -\pi < s_1 < ... < s_r < \pi$ on $[-\pi, \pi]$;

4) $\left\{\frac{h_k}{2\pi}+\frac{1}{q\left(s_k\right)}\right\}_{k=0}^{r} \cap Z=\{\emptyset\}$, where $h_k=\theta\left(s_k+0\right)-\theta\left(s_k-0\right),\ k=\overline{1,r};\ h_0=\theta(-\pi)-\theta(\pi)$.

It is required to find a piecewise-analytic function $F^{\pm}(z)$ on C with a section Γ, satisfying the conditions:

a) $F^{+}\in H_{pt}^{+}:\ 0<p^{-}\leq p^{+}<+\infty$;

b) $F^{-}\in\ {}_mH_{pt}^{-}$;

c) non-tangential boundary values of $F^{\pm}\left(e^{it}\right)$ on a unit circle Γ a.e. satisfy the relation:

$$F^{+}\left(e^{it}\right)+G(t)\,F^{-}\left(e^{it}\right)=g(t),\ \text{a.e. } t\in(-\pi,\pi),$$

where $g\in L_{pt}$ is an arbitrary function.

When summability indices are constant, the theory of these problems were sufficiently well studied (see. [6]). Let's consider Riemann's following homogeneous problem:

$$\left\{\begin{array}{c} F^{+}(\tau)+G(\tau)\,F^{-}(\tau)=0,\ \tau\in\Gamma; \\ F^{+}\in H_{pt}^{+};F^{-}\in\ {}_mH_{pt}^{-}. \end{array}\right. \tag{4}$$

Let's introduce into consideration the following analytic functions $X_i^{\pm}(z)$ interior (the sign " + ") and exterior (the sign " − ") to a unit circle.

$$X_1^{\pm}(z)\equiv\exp\left\{\pm\frac{1}{4\pi}\int_{-\pi}^{\pi} ln\left|G\left(e^{it}\right)\right|\frac{e^{it}+z}{e-z}dt\right\},$$

$$X_2^{\pm}(z)\equiv\exp\left\{\pm\frac{1}{4\pi}\int_{-\pi}^{\pi}\theta(t)\,\frac{e^{it}+z}{e-z}dt\right\}.$$

Let

$$Z_i(z)\equiv\left\{\begin{array}{cc} X_i^{+}(z), & |z|<1, \\ \left[X_i^{-}(z)\right]^{-1}, & |z|<1. \end{array}\right.$$

Denote $Z^{\pm}(z)\equiv Z_1^{\pm}(z)\cdot Z_2^{\pm}(z)$. Define $\{n_i\}_{i=1}^{r}\subset Z$ from the inequalities

$$\left\{\begin{array}{c} -\frac{1}{q\left(s_k\right)}<\frac{h_k}{2\pi}+n_k-n_{k-1}<\frac{1}{p\left(s_k\right)},\ k=\overline{1,r}; \\ n_0=0. \end{array}\right.$$

Let $\omega_r=\frac{h_0}{2\pi}+n_r$. Eearlier we proved

Theorem [9]. *Let $p\in H^{\ln}$, $1<p^{-}$; the conditions 1)-4) be fulfilled. Then, if it holds $-\frac{1}{q^{\pi}}<\omega_r<\frac{1}{p^{\pi}}$, the general solution of homogeneous problem (4) in the classes $\left(H_{pt}^{+};_m H_{pt}^{-}\right)$ is of the form $F(z)\equiv Z(z)\cdot P_m(z)$, where $P_m(z)$ is an arbitrary polynomial of power $\leq m$.*

Corollary. *Let all the requirements of the previous theorem be fulfilled. Then, provided $F^{-}(\infty)=0$ the Riemann's homogeneous problem (4) in the classes $\left(H_{pt}^{+};_m H_{pt}^{-}\right)$ has only a trivial solution, i.e. zero solution.*

Now, let's consider Riemann's homogeneous problem

$$\left\{\begin{array}{c} F^{+}(\tau)+G(\tau)\,F^{-}(\tau)=g(\tau),\ \tau\in\Gamma; \\ F^{+}\in H_{pt}^{+};\ F^{-}\in\ {}_mH_{pt}^{-}, \end{array}\right. \tag{5}$$

where $g(\tau)\in L_{pt}$ is an arbitrary function. Obviously, the problem (5) has a unique solution (if it is solvable) iff the appropriate problem (4) has only a trivial solution. In the general case the solution of problem (5) is of the form $F(z)=F_0(z)+Z(z)\cdot P_m(z)$, where $F_0(z)$ is one of the particular solutions of problem (5), $P_m(z)$ is a polynomial of power $\leq m$.

5. Main results

As $G(\tau)$ we take the concrete function $G\left(e^{it}\right) = e^{2i\alpha t}$, $t \in [-\pi, \pi]$. Suppose $\alpha \in R$. The complex case is similarly investigated.

At first we assume that $g\left(e^{it}\right)$ is a Holder function $[-\pi, \pi]$. Solve the problem (5) by the method devoloped in the monograph [8]. We get the particular solution $F_0(z)$ of the form:

$$F_0^+(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\alpha\theta} g\left(e^{i\theta}\right) d\theta}{\left(1+e^{i\theta}\right)^{2\alpha}\left(1 - z\cdot e^{-i\theta}\right)} (1+z)^{2\alpha},$$

$$F_0^-(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\alpha\theta} g\left(e^{i\theta}\right) d\theta}{\left(1+e^{i\theta}\right)^{2\alpha}\left(1 - z\cdot e^{-i\theta}\right)} z^{-2\alpha} (1+z)^{2\alpha}.$$

The fact that $F_0(z)$ satisfies the relation (5) follows directly from the Sokhotsky-Plamel formula. Denote

$$h_n^+(t) = \frac{\left(1+e^{it}\right)^{-2\alpha}}{2\pi} e^{i\alpha t} \cdot \sum_{k=0}^{n} C_{2\alpha}^{n-k} \cdot e^{ikt},\ n = \overline{0,\infty};$$

$$h_m^-(t) = -\frac{\left(1+e^{it}\right)^{-2\alpha}}{2\pi} e^{i\alpha t} \cdot \sum_{k=0}^{m} C_{2\alpha}^{m-k} \cdot e^{-ikt},\ m = \overline{1,\infty},$$

where $C_\beta^n = \dfrac{\beta(\beta-1)\ldots(\beta-n+1)}{n!}$ are binomial coefficients. Expanding the functions $F_0^+(z)$ and $F_0^-(z)$ respectively, in the vicinities of zero and a point at infinity in power of z, we get

$$F_0^+(z) = \sum_{n=0}^{\infty} a_n^+ \cdot z^n,\quad F_0^-(z) = \sum_{n=1}^{\infty} a_n^- \cdot z^{-n},$$

where

$$a_n^+ = \int_{-\pi}^{\pi} g\left(e^{i\theta}\right) \overline{h_n^+(\theta)} d\theta,\quad n \geq 0;\ a_m^- = \int_{-\pi}^{\pi} g\left(e^{i\theta}\right) \overline{h_n^-(\theta)} d\theta,\quad m \geq 1.$$

Let $|2\alpha| < 1$. It is easy to see that $F_0^+ \in H_1^+$; $F_0^- \in_{-1} H_1^-$. The relations [6]

$$\int_{-\pi}^{\pi} \left|F_0^+\left(e^{it}\right) - F_0^+\left(re^{it}\right)\right| dt \to 0,\ r \to 1-0;$$

$$\int_{-\pi}^{\pi} \left|F_0^-\left(e^{it}\right) - F_0^-\left(re^{it}\right)\right| dt \to 0,\ r \to 1+0,$$

yield

$$a_n^+ = \frac{1}{2\pi}\int_{-\pi}^{\pi} F_0^+\left(e^{it}\right) e^{-int} dt,\ \forall n \geq 0;\ a_m^- = \frac{1}{2\pi}\int_{-\pi}^{\pi} F_0^-\left(e^{it}\right) e^{imt} dt,\ \forall m \geq 1.$$

Using the representation of the Cauchy type integral with power character peruliarity in the vicinity of a discontinuity point of first order density (see [8], p.74), it is easy to show that if the conditions $0 < 2\alpha < 1$ and $g_1(1) = g(-1) = 0$ hold, the functions $F_0^\pm(\tau)$ are continuous on a unit circle. Therefore, the Fourier series of these functions by the system of exponents $\left\{e^{int}\right\}_{n\in Z}$ converge to them on $[-\pi, \pi]$ uniformly, since they satisfy some Holderian conditions on Γ. As the result we get

$$F_0^+\left(e^{it}\right) = \sum_{n=0}^{\infty} a_n^+ e^{int};\quad F_0^-\left(e^{it}\right) = \sum_{n=1}^{\infty} a_n^- e^{-int},$$

uniformly on $[-\pi,\pi]$. Considering these relations in (5) we get (where $g(\tau) = f(\tau)\cdot e^{i\alpha t}$, $\tau = e^{it}$; $f\left(e^{it}\right)$ is a Holder function on $[-\pi,\pi]$)

$$f\left(e^{it}\right) = \sum_{n=0}^{\infty} a_n^+ e^{i(n-\alpha)t} + \sum_{n=1}^{\infty} a_n^+ e^{-i(n-\alpha)t},$$

uniformly on $[-\pi,\pi]$. It is proved in [4] that for $|\alpha| < \frac{1}{2}$ the following relations

$$\left.\begin{array}{l} \int\limits_{-\pi}^{\pi} e^{i(n-\alpha)t}\overline{h_m^+(t)}dt = \delta_{nm},\ \forall n,m \geq 0;\ \int\limits_{-\pi}^{\pi} e^{i(n-\alpha)t}\overline{h_m^-(t)}dt = 0, \forall n \geq 0; \forall m \geq 1; \\ \int\limits_{-\pi}^{\pi} e^{-i(n-\alpha)t}\overline{h_m^+(t)}dt = 0,\ \forall n \geq 1; \forall m \geq 0;\ \int\limits_{-\pi}^{\pi} e^{-i(n-\alpha)t}\overline{h_m^-(t)}dt = \delta_{nm}, \forall n,m \geq 1. \end{array}\right\} \quad (6)$$

are fulfilled.

It directly follows from the Property A that, if $p(t) \in H^{\ln}$ and $p^- > 1$, then the system (1) belongs to L_{p_t}. In this case the space L_{q_t} is a space conjugated to L_{p_t} (see [16]). Consequently, it follows from statement 1 and representations for $h_n^{\pm}(t)$ that for $\alpha < \frac{1}{2q^{\pi}}$ the system $\{h_n^+; h_m^-\}$ belongs to L_{q_t}. Then, from relations (6) we get that while fulfilling the conditions formulated above, the system (1) and $\{h_n^+; h_m^-\}$ are conjugated and so (1) is minimal in L_{p_t}. Having paid attention to the Property B we get that for $\frac{1}{2} > \alpha \geq 0$ the system (1) is complete in L_{p_t}. Thus, if the inequality $0 \leq \alpha < \frac{1}{2q^{\pi}}$ is fulfilled, then (1) is complete and minimal in L_{p_t}.

Denote

$$I(z) = \int\limits_{-\pi}^{\pi} \frac{e^{i\alpha\theta} g_0(\theta)\, d\theta}{\left(1+e^{i\theta}\right)^{2\alpha}\left(1 - ze^{-i\theta}\right)},\ g_0(\theta) = g_0\left(e^{i\theta}\right).$$

Then we can represent $F_0^{\pm}(z)$ in the form

$$\left.\begin{array}{l} F_0^+(z) = \frac{1}{2\pi} I(z)(1+z)^{2\alpha},\ |z| < 1; \\ F_0^-(z) = \frac{1}{2\pi} I(z)\left(1+z^{-1}\right)^{2\alpha},\ |z| > 1. \end{array}\right\} \quad (7)$$

From the same reasonings we get that for finite functions $g_0(\theta)$ on $[-\pi,\pi]$, the Fourier series for boundary values $I^{\pm}\left(e^{i\theta}\right)$ converge to them uniformly on $[-\pi,\pi]$. Therewith, if $2\alpha > -\frac{1}{p^{\pi}}$, the functions $\left(1+e^{i\theta}\right)^{2\alpha}$ and $\left(1+e^{-i\theta}\right)^{2\alpha}$ belong to the space L_{p_t} and by the results of the paper [5], the Fourier series of these functions converge to them in L_{p_t}. Again, it follows from the Property B that for $-\frac{1}{2p^{\pi}} < \alpha < \frac{1}{2}$ the system (1) is complete in L_{p_t}. Combining the obtained results we arrive at the following conclusion.

Statement 2. *Let $p(t) \in H^{\ln}$, $p^- > 1$, and the inequality*

$$-\frac{1}{2p^{\pi}} < \alpha < \frac{1}{2q^{\pi}}, \quad (8)$$

be fulfilled. Then the system (1) is complete and minimal in L_{p_t}.

Now we study the basicity. Let (8) be fulfilled. Then the system (1) is minimal in L_{p_t} and let $\{h_n^+(t); h_m^-(t)\}_{n\geq 0; m\geq 1}$ be an appropriate conjugated system. Hence $\forall f \in L_{p_t}$ and consider the partial sum S_m:

$$S_m[f] = \sum_{n=0}^{m} a_n^+ e^{i(n-\alpha)t} + \sum_{n=1}^{m} a_n^- e^{-i(n-\alpha)t},$$

where

$$a_n^+ = \int_{-\pi}^{\pi} f(t)\,\overline{h_n^+(t)}dt, \quad n \geq 0; \; a_k^- = \int_{-\pi}^{\pi} f(t)\,\overline{h_n^-(\theta)}dt, \quad k \geq 1.$$

Let's consider the problem (5), where as the right hand side of $g(\tau)$ we take the function $g\left(e^{i\theta}\right) = e^{i\alpha t} f(\vartheta)$, furthermore, require $F^-(\infty) = 0$. Then, as it follows from Corollary 1, the problem (5) has a unique solution $F_0^{\pm}(z)$ in the classes $\left(H_{p_t}^+ ;_{-1} H_{p_t}^-\right)$ and thus $F_0^{\pm}\left(e^{it}\right) \in L_{p_t}$.

Show that

$$\sup_{\substack{m \\ \|f\|_{p_t}=1}} \|S_m[f]\|_{p_t} < +\infty.$$

As we have already seen

$$a_n^+ = \frac{1}{2\pi}\int_{-\pi}^{\pi} F_0^+\left(e^{it}\right) e^{-int}dt, \; \forall n \geq 0; \; a_k^- = \frac{1}{2\pi}\int_{-\pi}^{\pi} F_0^-\left(e^{it}\right) e^{ikt}dt, \; \forall k \geq 1.$$

We have

$$\|S_m[f]\|_{p_t} \leq \left\| e^{-i\alpha t}\sum_{n=0}^{m} a_n^+ e^{int}\right\|_{p_t} + \left\| e^{i\alpha t}\sum_{n=1}^{m} a_n^- e^{-int}\right\|_{p_t}.$$

Since the classic system of exponents $\left\{e^{int}\right\}_{n\in Z}$ forms a basis in L_{p_t} (see[5]),then considering the Property A hence we get

$$\|S_m[f]\|_{p_t} \leq M_1 \left\|F_0^+\left(e^{it}\right)\right\|_{p_t} + M_2\left\|F_0^-\left(e^{it}\right)\right\|_{p_t},$$

where $M_i, \; i = 1, 2$ are some constants. Applying the Sokhotsky-Plamel formula to the expressions $F_0^+(z)$ and $F_0^-(z)$ we get

$$F_0^+\left(e^{i\theta}\right) = ie^{i\alpha\theta} f(\theta) + S^+(f), \; F_0^-\left(e^{i\theta}\right) = ie^{-i\alpha\theta} f(\theta) + S^-(f),$$

where $S^{\pm}(f)$ are appropriate singular type integrals

$$S^+(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\alpha\theta} f(\theta)\,d\theta}{\left(1+e^{i\theta}\right)^{2\alpha}\left(1-e^{i(s-\theta)}\right)}\cdot\left(1+e^{is}\right)^{2\alpha},$$

$$S^-(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\alpha\theta} f(\theta)\,d\theta}{\left(1+e^{i\theta}\right)^{2\alpha}\left(1-e^{i(s-\theta)}\right)}\cdot\left(1+e^{-is}\right)^{2\alpha}.$$

Then, having paid attention to the Statement [11] we get that the integral operators $S^+(f)$ and $S^-(f)$ act boundedly from L_{p_t} to L_{p_t}, i.e.

$$\left\|S^{\pm}(f)\right\|_{p_t} \leq M\|f\|_{p_t}, \; \forall f \in L_{p_t}.$$

As the result we have

$$\|S_m[f]\|_{p_t} \leq M_1\left(M_3\|f\|_{p_t} + \left\|S^+(f)\right\|_{p_t}\right) +$$

$$+M_2\left(M_4\|f\|_{p_t} + \left\|S^-(f)\right\|_{p_t}\right) \leq M_5\|f\|_{p_t}, \; \forall f \in L_{p_t},$$

where $M_i, i = \overline{3,5}$ are some constants.

As the result, it follows from the basicity criterium that the system (1) forms a basis in L_{p_t}, i.e. the following theorem is valid.

Theorem 1. *Let $p(t) \in H^{\ln}, \; p^- > 1$, and the inequality $-\frac{1}{2p^{\pi}} < \alpha < \frac{1}{2q^{\pi}}$ be fulfilled. Then the system of exponents (1) forms a basis in L_{p_t}.*

Separately we consider the case $-\dfrac{1}{2p_\pi} \le \alpha \le -\dfrac{1}{2p^\pi}$. In this case, it follows from relations (6) and from expressions for $h_n^{\pm}(t)$ that the system (1) is minimal in L_{p_t}, since it has a biorthogonal system.Represent the system (1) in the form:

$$\left\{ e^{i[(n+1-(\alpha+1)]t}; e^{-i(m-\alpha)t} \right\}_{n\ge 0; m\ge 1}. \tag{9}$$

Multiplication of each term of the system (9) by the function $e^{i\frac{t}{2}}$ doesn't influence on its completeness in L_{p_t}. As the result we get the system $\left\{ I_{n;m}^{\widetilde{\alpha}}(t) \right\}_{n\ge 1; m\ge 1}$, where $I_{n;m}^{\widetilde{\alpha}}(t) \equiv \left(e^{i(n-\widetilde{\alpha})\cdot t}, e^{-i(m-\widetilde{\alpha})\cdot t} \right)$, $\widetilde{\alpha} = \alpha + \dfrac{1}{2}$. It is easy to notice that $\dfrac{1}{p^\pi} + \dfrac{1}{q_\pi} = 1$; $\dfrac{1}{p_\pi} + \dfrac{1}{q^\pi} = 1$.Therefore, the inequality $\dfrac{1}{2q^\pi} \le \widetilde{\alpha} \le \dfrac{1}{2q_\pi} < \dfrac{1}{2}$ is fulfilled for $\widetilde{\alpha}$. Then by the previous results we get that the system $\left\{ I_{n;m}^{\widetilde{\alpha}}(t) \right\}_{n\ge 0; m\ge 1}$ is complete in L_{p_t}. It follows from the expressions for $\{h_n^{\pm}(t)\}$ and from Statement 1 that in this case the system $\{h_n^{\pm}(t)\}$ doesn't belong to the space L_{q_t}. Since the system $\left\{ I_{n;m}^{\widetilde{\alpha}}(t) \right\}_{n\ge 1; m\ge 1}$ is complete in L_{p_t} then from the uniqueness of biorthogonal system to the complete system we get that $\left\{ I_{n;m}^{\widetilde{\alpha}}(t) \right\}_{n\ge 0; m\ge 1}$ is not minimal in L_{p_t} and as a result of that the system $\left\{ I_{n;m}^{\widetilde{\alpha}}(t) \right\}_{n;m\ge 1}$ and so the system (1) is complete and minimal in L_{p_t}. The fact that the system (1) doesn't form a basis in L_{p_t} is proved similar to the paper [13]. We arrive at the following conclusion: if $-\dfrac{1}{2p_\pi} \le \alpha \le -\dfrac{1}{2p^\pi}$, the system (1) is complete and minimal in L_{p_t}. And now, let $\alpha < -\dfrac{1}{2p_\pi}$, for example $-\dfrac{1}{2p_\pi} - \dfrac{1}{2} \le \alpha < -\dfrac{1}{2p_\pi}$. In this case, it holds $-\dfrac{1}{2p^\pi} \le \widetilde{\alpha} < \dfrac{1}{2q^\pi}$ and so the system $\left\{ I_{n;m}^{\widetilde{\alpha}}(t) \right\}_{n\ge 0; m\ge 1}$ is complete, and minimal in L_{p_t}. As the result the system (1) is not complete, but minimal in L_{p_t}. In the similar way we show that for $\alpha \ge \dfrac{1}{2q^\pi}$ the system is complete, but not minimal in L_{p_t}.

Combining all the obtained results, we have the following theorem.

Theorem 2. *Let $p(t) \in H^{\ln}$, $p^- > 1$. The system (1) is complete in L_{p_t} iff $\alpha \ge -\dfrac{1}{2p_\pi}$; it is minimal in L_{p_t} only for $\alpha < \dfrac{1}{2q^\pi}$.*

Let the inequality $\alpha < \dfrac{1}{2q^\pi}$ hold. By theorem 2, in this case the system (1) is minimal in L_{p_t}. It directly follows from analytical expressions for the conjugated system $\{h_n^{\pm}(t)\}$ that

$$h_0^+(t) = \frac{1}{2\pi} \cdot \frac{e^{i\alpha t}}{\left(1+e^{it}\right)^{2\alpha}}.$$

We have

$$\overline{c_0^+} = \int_{-\pi}^{\pi} \overline{h_0^+(t)} dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{dt}{\left(1+e^{-it}\right)^{2\alpha} \cdot \left(e^{it}\right)^{\alpha}} =$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{dt}{\left(e^{i\frac{t}{2}} + e^{-i\frac{t}{2}}\right)^{2\alpha}} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{dt}{\left(2\cos\dfrac{t}{2}\right)^{2\alpha}} \neq 0.$$

We consider the system $\{H_n^+; H_m^-\}_{n\ge 0; m\ge 1}$

$$H_0^+ = \frac{1}{c_0^+} h_0^+;\ H_n^{\pm} = h_n^{\pm} - \frac{c_n^{\pm}}{c_0^+} h_0^+, \tag{10}$$

where $c_n^{\pm} = \int\limits_{-\pi}^{\pi} h_n^{\pm}(t)\,dt, \quad \forall n \geq 1$. It is easy to verify that the systems $\left\{H_n^+; H_{n+1}^-\right\}_{n\geq 0}$ and (2) are biorthonormed. Thus, for $\alpha < \dfrac{1}{2q^{\pi}}$ the system (2) is minimal in L_{p_t}. The remaining cases for the values of α are similarly proved.

Let $-\dfrac{1}{2p^{\pi}} < \alpha < \dfrac{1}{2q^{\pi}}$. Take $\forall f \in L_{p_t}$ and consider

$$S_m^0[f] = f_0^+ + \sum_{n=1}^{m}\left[f_n^+ e^{-i\alpha t}e^{int} + f_n^- e^{i\alpha t}e^{-int}\right],$$

where $f_n^{\pm}$ are biorthogonal coefficients of the function f by the system (2). Considering expression (10) for $H_n^{\pm}$ it is easy to show that $\|S_m^0(f) - f\|_{p_t} \to 0, \ m \to \infty$. This proves the basicity of the system (2) in the considered case. Thus, it is proved.

Theorem 3. *Let $p(t) \in H^{\ln}$, $p^- > 1$. The system (2) forms a basis in L_{p_t} iff $-\dfrac{1}{2p^{\pi}} < \alpha < \dfrac{1}{2q^{\pi}}$. Moreover, it is complete in L_{p_t} only for $\alpha \geq -\dfrac{1}{2p_{\pi}}$; it is minimal iff $\alpha < \dfrac{1}{2q^{\pi}}$. For $-\dfrac{1}{2p_{\pi}} \leq \alpha \leq -\dfrac{1}{2p^{\pi}}$ it is complete and minimal, but doesn't form a basis in L_{p_t}.*

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